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One-loop conformal anomalies from AdS/CFT in the Schrödinger representation

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Abstract: We compute the conformal anomalies of boundary CFTs for scalar and fermionic fields propagating in AdS spacetime at one-loop. The coefficients are quantized, with values related to the mass-spectra for Kaluza-Klein compactifications of Supergravity on $AdS_5 \times S^5$ and $AdS_7 \times S^4$. Our approach interprets the partition function of fields on AdS spacetime in terms of wave-functionals that satisfy the functional Schrödinger equation.

Keywords: Anomalies in Field and String Theories, Conformal Field Models in String Theory.

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Contents

1. Introduction

Given the current interest in supergravity in an anti de Sitter background it would seem useful to develop techniques for studying quantum field theory beyond tree-level in such a spacetime. In this paper we will consider the simplest one-loop quantity, namely the conformal anomaly of the boundary theory. The central object of study in the AdS/CFT correspondence, [1] is a functional integral for a quantum field theory in Anti de Sitter space expressed in terms of the boundary values of the field. Whilst this can be treated by the usual semiclassical expansion it may also be interpreted as the *large* time limit of the Schrödinger functional of the quantum field theory and so satisfies a functional Schrödinger equation. We will show how to obtain this large time behaviour from a short time expansion using analyticity. We will illustrate our method by reproducing the known two-point functions of the CFT for free scalar and fermionic fields in AdS and then derive the one-loop contribution to the conformal anomalies for these theories. Our approach also yields a straightforward calculation of the tree-level contribution to the conformal anomaly for the pure gravity sector.

Following [2] we consider the euclidean version of AdS_{d+1} with co-ordinates $\{x^\mu\} \equiv \{t, x^1, \ldots, x^d\}$ and metric

$$ds^2 = \frac{1}{(x^0)^2} \sum_{\mu=0}^{d} \big(dx^\mu\big)^2 = \frac{1}{t^2}\Big(dt^2 + dx \cdot dx\Big). \tag{1.1}$$

We will think of t, which is restricted to the range $t > 0$, as euclidean time. The boundary, ∂M, consists of R^d at $t = 0$ conformally compactified to a sphere by adding a point corresponding to $t = \infty$ where the metric vanishes. For illustration consider a scalar field theory propagating on this space-time. We study the functional integral

$$Z[\varphi] = \int \mathcal{D}\phi e^{-S} \mid_{\phi|\partial M=\varphi}, \qquad S = \frac{1}{2}\int d^{d+1}x\,\sqrt{g}\Big(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\Big), \tag{1.2}$$

where $\mathcal{D}\phi$ is the volume element induced by the reparametrization invariant inner product on variations of ϕ, $\|\delta\phi\|^2 = \int dt\,dx\,\delta\phi^2/t^{d+1}$. We will need to regulate this by restricting t to the range $\tau > t > \tau'$, so define

$$\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi] = \int \mathcal{D}\phi e^{-S} \mid_{\phi(\tau)=\tilde{\varphi},\phi(\tau')=\varphi} . \tag{1.3}$$

Since the point corresponding to $t = \infty$ is part of ∂M we set $\tilde{\phi} = \lim_{|x|\to\infty}\varphi(x)$ as we take the limit of large τ and small τ' to recover Z. This is usually described as a partition function, but it may also be interpreted in terms of the wave-functionals that represent states in the Schrödinger representation. First change variables from t, ϕ to $\bar{t} = \ln t, \bar{\phi} = \phi/t^{d/2}$ so that the volume element and kinetic term become the usual ones associated with the canonical quantization of $\bar{\phi}$. Thus $Z[\varphi]$ is the $\tau' \to 0, \tau \to \infty$ limit of

$$\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi] = \int \mathcal{D}\bar{\phi}e^{-\bar{S}-S_b} \equiv \bar{Z}\Big[\bar{\phi}_f,\bar{\phi}_i\Big]e^{-S_b}, \tag{1.4}$$

where

$$\bar{S} = \frac{1}{2}\int d\bar{t}\,dx\left(\left(\frac{\partial\bar{\phi}}{\partial\bar{t}}\right)^2 + \frac{d^2}{4}\bar{\phi}^2 + t^2\nabla\phi\cdot\nabla\phi + t^{-d}V\Big(\bar{\phi}t^{d/2}\Big)\right), \tag{1.5}$$

$$S_b = \frac{d}{4}\Big(\bar{\phi}_f^2 - \bar{\phi}_i^2\Big) \tag{1.6}$$

and $\bar{\phi}_f$, $\bar{\phi}_i$ are the value of $\bar{\phi}$ on the surfaces $\bar{t} = \bar{t}_1 = \ln\tau$ and $\bar{t} = \bar{t}_2 = \ln\tau'$ respectively. Now $\bar{Z}[\bar{\phi}_f,\bar{\phi}_i]$ can be interpreted as the Schrödinger functional, i.e., the matrix element of the time evolution operator between eigenstates of the field operator,

$$\bar{Z}\Big[\bar{\phi}_f,\bar{\phi}_i\Big] = \left\langle\bar{\phi}_f\middle|T\exp\left(-\int_{\bar{t}_2}^{\bar{t}_1}dt H(t)\right)\middle|\bar{\phi}_i\right\rangle, \tag{1.7}$$

which satisfies the functional Schrödinger equation

$$\begin{aligned}\frac{\partial}{\partial\bar{t}_1}\bar{Z}\Big[\bar{\phi}_f,\bar{\phi}_i\Big] = -\frac{1}{2}\int dx\bigg(-\frac{\delta^2}{\delta\bar{\phi}_f^2} + t^2\nabla\bar{\phi}_f\cdot\nabla\bar{\phi}_f + \frac{d^2}{4}\bar{\phi}_f^2 + {}\\ + t^{-d}V\Big(\bar{\phi}_f t^{d/2}\Big)\bigg)\bar{Z}\Big[\bar{\phi}_f,\bar{\phi}_i\Big],\end{aligned} \tag{1.8}$$

with the initial condition that it tends to a the delta-functional $\delta[\bar{\phi}_f - \bar{\phi}_i]$ as $\bar{t}_1$ approaches $\bar{t}_2$. We can re-write this in terms of the boundary values of our original variables t, ϕ, i.e., τ and $\tilde{\varphi}$

$$\frac{\partial}{\partial\tau}\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi] = -\frac{1}{2}\int dx\left(-\Omega^{-1}\frac{\delta^2}{\delta\tilde{\varphi}^2} + \Omega\nabla\tilde{\varphi}\cdot\nabla\tilde{\varphi} + \Omega' V(\tilde{\varphi}) + \frac{\mathcal{E}}{\tau}\right)\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi]\,, \tag{1.9}$$

where $\Omega = \tau^{1-d}$, $\Omega' = \Omega/\tau^2$. $\mathcal{E}$ arises from the action of the laplacian on S_b, formally

$$\mathcal{E} = -\tau^d\frac{\delta^2}{\delta\tilde{\varphi}^2}S_b = -\frac{d}{2}\delta^d(0)\,. \tag{1.10}$$

Clearly the coincident functional derivatives stand in need of regularization, which introduces a short-distance cut-off. By extending the flat-space arguments of Symanzik [3] we would expect that for a renormalizable field theory wave-function renormalization and an appropriate choice of the dependence of V on the cut-off would ensure the finiteness of the solution to (1.9) in the limit that the cut-off is removed. Since this can involve the use of counter-terms associated with the boundaries we would expect that the renormalization constants may depend on τ and τ'. However in many applications the tree-level solution is sufficent for which these considerations are unnecessary. We will see later that $\mathcal{E}$ contributes to the conformal anomaly. A similar argument yields the Schrödinger equation that gives the τ' dependence

$$-\frac{\partial}{\partial\tau'}\,\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi] = -\frac{1}{2}\int dx\left(-\Omega^{-1}\frac{\delta^2}{\delta\varphi^2} + \Omega\nabla\varphi\cdot\nabla\varphi + \Omega' V(\varphi) - \frac{\mathcal{E}}{\tau}\right)\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi]\,, \tag{1.11}$$

In [4, 5, 6, 7] a general approach to solving the functional Schrödinger equation was developed. Consider the logarithm of $\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi]$, $W_{\tau,\tau'}[\tilde{\varphi},\varphi]$. In perturbation theory this is a sum of connected Feynman diagrams. In Quantum Mechanics we might try to solve the Schrödinger equation by expanding in a suitable basis of functions. An analogous construction is to expand in terms of local functionals, i.e., a derivative expansion. Now $W_{\tau,\tau'}[\tilde{\varphi},\varphi]$ is nonlocal, but it will reduce to the integral of an infinite sum of local terms when it is evaluated for fields that vary slowly on the scale of τ. Each term will depend on a finite power of the field and its derivatives at a single point, $\mathbf{x}$, on the quantization surfaces $t=\tau, t=\tau'$. Although this expansion is appropriate for slowly varying fields the functional for arbitrarily varying fields can be reconstructed from it because the functional evaluated for scaled fields $\tilde{\varphi}(x/\sqrt{\rho})$ and $\varphi(x/\sqrt{\rho})$ can be analytically continued to the complex ρ-plane with the negative real axis removed. Thus Cauchy's theorem will allow us to relate rapidly varying fields (small ρ) to slowly varying ones (large ρ), and from this we can use the behaviour for small τ to obtain that for large τ, which is what is needed in the AdS/CFT correspondence. Similar considerations enable the Schrödinger

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equation, which because it has an ultra-violet cut off involves rapidly varying fields, to be turned into an equation acting directly on the local expansion, determining the coefficients of that expansion from a set of *algebraic* equations. (In flat space these equations can be solved perturbatively where they yield the usual results for short-distance effects in scalar field theory [5] and for the beta-function in Yang-Mills theory [7].)

In the next section we prove the analyticity property that we will need to relate short and large time behaviour. We apply this to the example of the scalar field in section 3, reproducing the known scaling property of the two-point function in the CFT, and in section 4 we compute the one-loop conformal anomaly for this d-dimensional boundary theory, showing that it vanishes unless $\sqrt{(d/2)^2+m^2}$ is a positive integer, where m is the mass in the AdS lagrangian. For $d=2$ this integer is the value of the central charge of the Virasoro algebra. The free fermion is discussed in section 5 where we show that the chiral projection that is known to occur in this theory is related to the reducibility of the Floreanini-Jackiw representation of fermionic fields. We also compute the conformal anomaly at one-loop for fermions and find that similarly to the scalar case, it vanishes unless $|m|+1/2$ is an integer. For $d=2$ this integer is again the value of the central charge of the Virasoro algebra. For $d=4$ these values of the masses correspond to the scalar and fermion mass spectra in the Kaluza-Klein compactification of Supergravity on $AdS_5\times S^5$. Finally, in section 6 we discuss pure gravity, for which our formalism yields a simple calculation of the conformal anomaly.

2. Analyticity of Schrödinger functional

To show analyticity of $\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi]$ we generalize to curved space the arguments of [4, 5, 6, 7]. We first make the dependence on $\tilde{\varphi},\varphi$ explicit by modifying a standard phase-space derivation of the functional integral representation of the Schrödinger functional. By the usual argument we can write $Z[\bar{\phi}_f,\bar{\phi}_i]$ as

$$\lim_{n\to\infty}\int\prod_{j=1}^{n}\mathcal{D}\bar{\phi}_j\prod_{j=1}^{n+1}\mathcal{D}\Pi_j\exp\left(\sum_{j=1}^{n+1}\Big(-H\Big[\Pi_j,\bar{\phi}_{j-1}\Big]\delta\bar{t}_j\Big)+i\int dx\Pi_j\Big(\bar{\phi}_j-\bar{\phi}_{j-1}\Big)\right),\tag{2.1}$$

where Π is the eigenvalue of the canonical momentum conjugate to $\bar{\phi}$, $\delta\bar{t}_j=\bar{t}_j-\bar{t}_{j-1}$ and $\bar{\phi}_{n+1}=\bar{\phi}_f$ and $\bar{\phi}_0=\bar{\phi}_i$. $\bar{\phi}_f$ appears in this expression only in the $i\Pi_{n+1}\bar{\phi}_{n+1}$ term in the exponent, whereas $\bar{\phi}_i$ appears in $-i\Pi_1\bar{\phi}_0$, and terms proportional to $\delta\bar{t}_1$, which can be neglected as $\delta\bar{t}_1\to 0$. So the contribution of $\bar{\phi}_i$ and $\bar{\phi}_f$ to (2.1) can be manifested by adding to the exponent $i\int dx(\Pi_{n+1}\bar{\phi}_f-\Pi_1\bar{\phi}_i)$ and taking $\bar{\phi}_{n+1}=\bar{\phi}_0=0$. Thus $\bar{\phi}_i$ and $\bar{\phi}_f$ appear as sources coupled to $\dot{\bar{\phi}}$ when we integrate out the momenta, Π_j, so that we arrive at the functional integral

$$\int \mathcal{D}\bar{\phi} \exp\Big(-\bar{S} + \int dx \Big(\bar{\phi}_f \dot{\bar{\phi}}(\bar{t}_1) - \bar{\phi}_i \dot{\bar{\phi}}(\bar{t}_2)\Big)\Big) \exp\Big(\int dx\, \Lambda \Big(\bar{\phi}_i^2 + \bar{\phi}_f^2\Big)\Big), \quad (2.2)$$

where the boundary condition on $\bar{\phi}$ is now that it should vanish at $\bar{t} = \bar{t}_1, \bar{t}_2$. Λ is a regularization of $1/\epsilon$ which cancels certain divergences that arise in the evaluation of (2.2) whose origin is explained in [4]. If we now interchange the rôles of $\bar{t}$ and x^1, and think of x^1 as euclidean time and $\bar{t}, x^2, \ldots, x^d$ as spatial co-ordinates then we can give an alternative interpretation of the functional integral (ignoring the Λ factor for the time being) as the vacuum expectation value

$$\Big\langle O_r \Big| T \exp\Big(\int dx^1 \varphi_i \hat{R}_i(x^1)\Big) \Big| O_r \Big\rangle, \quad (2.3)$$

where $|O_r\rangle$ is the vacuum for the hamiltonian, $\hat{H}_r$, associated with the quantization surfaces of constant x^1 and $\bar{t}_2 < \bar{t} < \bar{t}_1$. Unlike the previous hamiltonian which depended on $\bar{t}$, this operator is independent of its associated 'time', x^1. φ_i is a compact notation for the sources, so that

$$\varphi_i \hat{R}_i(x^1) \equiv \int dx^2 \cdots dx^d \Big(\bar{\phi}_f \dot{\bar{\phi}}(\bar{t}_1) - \bar{\phi}_i \dot{\bar{\phi}}(\bar{t}_2)\Big). \quad (2.4)$$

Expanding (2.3) in powers of the sources and using $\hat{H}_r$ to generate the x^1-dependence of the operators $\hat{R}_i$ gives

$$\begin{aligned} &\Big\langle O_r \Big| T \exp\Big(\int dx^1 \varphi_i \hat{R}_i(x^1)\Big) \Big| O_r \Big\rangle = \\ &\quad = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dx_n^1 \int_{-\infty}^{x_n^1} dx_{n-1}^1 \cdots \int_{-\infty}^{x_3^1} dx_2^1 \int_{-\infty}^{x_2^1} dx_1^1 \prod_{j=1}^{n} \varphi_{i_j}(x_j^1) \times \\ &\quad\quad \times \Big\langle O_r \Big| \hat{R}_{i_n}(0) e^{(x_{n-1}^1 - x_n^1)\hat{H}_r} \hat{R}_{i_{n-1}}(0) \cdots \hat{R}_{i_2}(0) e^{(x_2^1 - x_1^1)\hat{H}_r} \hat{R}_{i_1}(0) \Big| O_r \Big\rangle. \end{aligned} \quad (2.5)$$

We have taken the eigenvalue of $\hat{H}_r$ belonging to $|O_r\rangle$ to be zero. Fourier transforming the x^1-dependence of the sources as $\varphi_i(x^1) = \int dk\, \tilde{\varphi}_i(k) \exp(-ikx^1)$ enables the x^1 integrals to be done yielding

$$\begin{aligned} &\sum_{n=0}^{\infty} \delta\Big(\sum_{j=1}^{n} k_j\Big) \prod_{j=1}^{n} \int dk_j\, \tilde{\varphi}_{i_j}(k_j) \Big\langle O_r \Big| \hat{R}_{i_n}(0) \frac{1}{\hat{H}_r - i\sum_1^{n-1} k_j} \times \\ &\qquad\qquad \times \hat{R}_{i_{n-1}}(0) \cdots \hat{R}_{i_2}(0) \frac{1}{\hat{H}_r - ik_1} \hat{R}_{i_1}(0) \Big| O_r \Big\rangle. \end{aligned} \quad (2.6)$$

Suppose that we had computed the Schrödinger functional for new sources obtained by scaling x^1, $\varphi_i(x^1/\sqrt{\rho}, x^2, \ldots, x^d)$, with ρ real and positive. Then we would have obtained the same expression as (2.6) but multiplied by $\sqrt{\rho}$ and with the $\hat{H}_r$ in the denominators replaced by $\sqrt{\rho}\hat{H}_r$. We took ρ to be real and positive, but we can use this expression to continue to the complex ρ-plane. Since the eigenvalues of $\hat{H}_r$ are real we conclude that the result is analytic in the whole plane with the negative real axis removed. This assumes that we work to finite order in the sources, and that the spectral decomposition of (2.6) as a sum over eigenvalues of $\hat{H}_r$ converges, as we should expect if the Schrödinger functional is finite. The terms in Λ in (2.2) do not affect this conclusion. By repeating the argument with x^1 interchanged with each of the other co-ordinates in turn we conclude that for $\varphi_i^\rho(x) \equiv \varphi_i(x/\sqrt{\rho})$ the Schrödinger functional $Z[\bar{\phi}_f^\rho, \bar{\phi}_i^\rho]$ and consequently $\Psi_{\tau,\tau'}[\tilde{\varphi}^\rho, \varphi^\rho]$ are (to any finite order in the sources) analytic in ρ in the plane cut along the negative real axis. Since $x \to x/\sqrt{\rho}, t \to t/\sqrt{\rho}$ is an isometry of (1.1) it follows that

$$\Psi_{\tau,\tau'}\left[\tilde{\varphi}^\rho, \varphi^\rho\right] = \Psi_{\tau/\sqrt{\rho},\tau'/\sqrt{\rho}}\left[\tilde{\varphi}, \varphi\right] \tag{2.7}$$

and we see that analyticity in ρ corresponds to the analyticity in time associated with Wick rotation. Subject to the caveat that we work to finite order in the sources the logarithm $\Psi_{\tau,\tau'}[\tilde{\varphi}^\rho, \varphi^\rho]$, $W_{\tau,\tau'}[\tilde{\varphi}, \varphi]$, is just a sum of products of terms appearing in (2.6) so it too is analytic in the cut ρ-plane when it is evaluated for the scaled sources.

We can use (2.6) to justify a local expansion for $W_{\tau,\tau'}[\tilde{\varphi}, \varphi]$ with a non-zero radius of convergence. Since $\hat{H}_r$ is the hamiltonian for a field theory on a finite 'spatial' interval such that $\hat{\phi}$ vanishes at the ends there is a mass-gap of the order of $1/\tau$ for small τ. When the momenta k_j are sufficently small on the scale of this mass-gap we can expand the denominators in the spectral decomposition of (2.6) in integer powers of $\sum k$, except for the contribution of the vacuum, which is of the form $1/\sum k$. The singular behaviour as $k \to 0$ must disappear when we take the logarithm to ensure cluster decomposition. We conclude that any term of finite order in the sources in $W_{\tau,\tau'}[\tilde{\varphi}, \varphi]$ has a local expansion for $\tilde{\varphi}, \varphi$ that vary sufficently slowly with x. This will take the form $W_{\tau,\tau'}[\tilde{\varphi}, \varphi] = \int d^dx(a\varphi^2 + b\tilde{\varphi}^2 + c\varphi\nabla^2\varphi + d\varphi\nabla^2\nabla^2\varphi \cdots)$ with $a, b, c, \ldots$ depending on τ, τ'. If the Fourier transforms of the sources have bounded support then $\tilde{\varphi}^\rho, \varphi^\rho$ are slowly varying for large ρ. Since each term in the expansion of $W_{\tau,\tau'}[\tilde{\varphi}^\rho, \varphi^\rho]$ has a simple dependence on ρ, we have that for large ρ the functional $W_{\tau,\tau'}[\tilde{\varphi}^\rho, \varphi^\rho] = \sqrt{\rho}^d \int d^dx(a\varphi^2 + b\tilde{\varphi}^2 + c\varphi\nabla^2\varphi/\rho + d\varphi\nabla^2\nabla^2\varphi/\rho^2 \cdots$. The powers of ρ are $d/2+$ integer so we conclude that the cut on the negative real axis in $\sqrt{\rho}^d W_{\tau,\tau'}[\tilde{\varphi}^\rho, \varphi^\rho]$ runs only a finite distance from the origin.

We will now exploit the analyticity to obtain the logarithm of the partition function $\log Z[\varphi] = \lim_{\tau\to\infty,\tau'\to 0} W_{\tau,\tau'}[0, \varphi]$ from the small τ behaviour, which is computable from a power series solution of the Schrödinger equation. (We take $\tilde{\varphi} = \lim_{x\to\infty}\varphi = 0$.) For simplicity we assume continuity in τ' at the origin, and consider $W_{\tau/\sqrt{\rho},0}[\tilde{\varphi}, \varphi]$. In general there will need to be some τ-dependent renormalization in order that the limit as $\tau \to \infty$ exists, including wave-function renormalization, $\varphi_{\rm ren} = \sqrt{z(\tau)}\varphi$, this will be the case even for free 'massive' fields at tree-level. Given that φ is a scalar the isometry $x^\mu \to \lambda x^\mu$ implies that the functional takes the form

$$\int d^dx\,\frac{\Omega}{\tau}\left(a+\frac{1}{2}\tilde{\varphi}\Gamma\big(-\tau^2\nabla^2\big)\tilde{\varphi}+\tilde{\varphi}\Xi\big(-\tau^2\nabla^2\big)\varphi+\frac{1}{2}\varphi\Upsilon\big(-\tau^2\nabla^2\big)\varphi+\cdots\right), \quad (2.8)$$

where the dots stand for terms of higher order in the fields of which the general term is of the form

$$\frac{\Omega}{\tau}\int d^dx\,\Gamma_{m,n}\Big(\tau\nabla_1,\ldots,\tau\nabla_{n+m}\Big)\varphi(x_1)\cdots\varphi(x_n)\tilde{\varphi}(x_{n+1})\cdots\tilde{\varphi}(x_{n+m})\,|_{\{x_i=x\}}\,. \quad (2.9)$$

At short times, or equivalently, for slowly varying fields, we have the local expansions

$$\Gamma=\sum_{n=0}^{\infty}b_n\big(-\tau^2\nabla^2\big)^n,\quad \Xi=\sum_{n=0}^{\infty}c_n\big(-\tau^2\nabla^2\big)^n,\quad \Upsilon=\sum_{n=0}^{\infty}f_n\big(-\tau^2\nabla^2\big)^n, \quad (2.10)$$

with b_n, c_n, f_n constants. Renormalizability would imply that

$$\frac{1}{\tau^d z(\tau)}\Big[\Upsilon\big(-\tau^2\nabla^2\big)+\text{polynomial in}\nabla\Big] \quad (2.11)$$

is finite as $\tau\to\infty$. Suppose that for large τ the renormalization constant depends on τ as $z(\tau)\sim\tau^{2q}$ then finiteness of the limit of (2.11) requires that for large τ, $\Upsilon(-\tau^2\nabla^2)\sim(-\tau^2\nabla^2)^{d/2+q}\upsilon$, and our problem is to calculate υ and q. The general term in (2.9) should depend on τ as $\Gamma_{m,n}(\tau\nabla_1,\ldots,\tau\nabla_{n+m})\sim\tau^{d+(n+m)q}F_{m,n}(\nabla_1,\ldots,\nabla_{n+m})$ and we need to calculate $F_{m,n}$. Now our previous arguments imply that $\Upsilon(1/\rho)$ is analytic in the complex plane with a finite cut extending from the origin along the negative real axis so we can evaluate the following integral

$$I(\lambda)=\frac{1}{2\pi i}\int_C\frac{d\rho}{\rho}e^{\lambda\rho}\Upsilon\left(\frac{1}{\rho}\right) \quad (2.12)$$

in two ways. We take C to be a circle centred on the origin and large enough for us to be able to use the local expansions (2.10) to give

$$I(\lambda)=\sum_{n=0}^{\infty}\frac{f_n\lambda^n}{n!}\,. \quad (2.13)$$

The integral may also be evaluated by collapsing the contour C onto the cut. Let this consist of a small circle about the origin, of radius η, and two lines close to the negative real axis running from the circle to the end of the cut. The contribution from the latter is suppressed if the real part of λ is large and positive. That from the circle is controlled by the large time behaviour

$$\frac{1}{2\pi i}\int_{|\rho|=\eta}d\rho\frac{\upsilon e^{\lambda\rho}}{\rho^{d/2+q+1}}=\upsilon\lambda^{d/2+q}\left(\frac{1}{2\pi i}\int_{|\rho|=|\lambda|\eta}d\rho\frac{e^{\rho}}{\rho^{d/2+q+1}}\right) \quad (2.14)$$

and for large λ this is $\lambda^{d/2+q}/\Gamma(d+2q+1)$. So, as the real part of λ tends to $+\infty$ we obtain

$$I(\lambda) = \sum_{n=0}^{\infty} \frac{f_n \lambda^n}{n!} \sim \frac{\upsilon \lambda^{d/2+q}}{(d+2q)!} \tag{2.15}$$

enabling us to compute υ and q from a knowledge of the local expansion alone. Now for positive real λ, $\sum_{n=0}^{\infty} f_n \lambda^n$ is an alternating series with finite radius of convergence. By comparing terms with those of $\exp(-\mathrm{constant}\lambda)$ it follows that this converges for all λ and so we can take λ large, even though this series is in positive powers of λ. Furthermore, as we shall see in some examples below, we obtain a good approximation to the large λ limit by truncating the series at some order, and then taking λ as large as is consistent with the truncation, i.e., so that the term of highest order in λ is a small fraction of the sum. This generalizes in an obvious way to the other terms in (2.8).

3. Example: free scalar field

As an example consider the free massless theory so $V = 0$ in (1.2); the action is

$$S = \frac{1}{2} \int d^d x \, dt \left(\Omega \sum_{\mu=0}^{d} (\partial_\mu \phi)^2 \right) \tag{3.1}$$

with $\Omega = 1/t^{d-1}$. For this gaussian functional integral only those terms shown explicitly in (2.8) are present. Substituting this into the Schrödinger equation (1.9) yields

$$\frac{\partial}{\partial \tau}\left(\frac{\Omega}{\tau}\Gamma\right) = \frac{\Omega}{\tau^2}\Gamma^2 + \Omega\nabla^2 , \qquad \frac{\partial}{\partial \tau}\left(\frac{\Omega}{\tau}\Xi\right) = \frac{\Omega}{\tau^2}\Gamma\Xi, \qquad \frac{\partial}{\partial \tau}\left(\frac{\Omega}{\tau}\Upsilon\right) = \frac{\Omega}{\tau^2}\Xi^2 ,$$
$$\frac{\partial}{\partial \tau}\left(\frac{\Omega}{\tau}a\right) = \frac{1}{2\tau}\left(\Gamma + \frac{d}{2}\right)\delta^d(x-y)\Big|_{x=y} . \tag{3.2}$$

These, together with the initial condition, lead to the recursive solution of the coefficients of the local expansions (2.10)

$$b_0 = -d = -c_0 = f_0 , \qquad b_1 = \frac{-1}{2+d} , \qquad b_n = \frac{\sum_{q=1}^{n-1} b_q b_{n-q}}{2n+d} ,$$
$$c_n = \frac{\sum_{q=1}^{n} b_q c_{n-q}}{2n} , \qquad f_n = \frac{\sum_{q=0}^{n} c_q c_{n-q}}{2n-d} . \tag{3.3}$$

If we were to take d to be an even positive integer the relations for the f_n would break down, thus keeping d variable regulates the solution for f_n. Note that there is no need to solve these relations further as they are ideally suited to numerical evaluation of the coefficients.

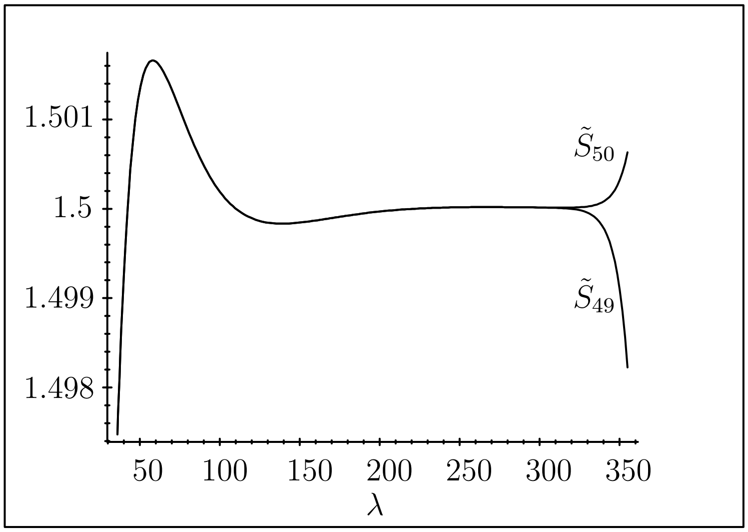

Figure 1: The truncated series as a function of λ.

To illustrate the calculation of υ and q take the example of $d = 3$. Our discussion will involve numerical computations. Although the rest of this paper is concerned with analytic results the numerical work of this section highlights the use of the derivative expansions and is readily generalisable to interacting theories [5]. From (2.15) we have that for large λ, $\tilde{I}(\lambda) \equiv d(\log(I(\lambda))/d(\log(\lambda)) \to \tilde{d}/2 + q$. Truncating the infinite series to its first N terms, $S_N(\lambda)$, gives an approximation to this. In figure 1 we have shown $\tilde{S}_N \equiv d(\log(S_N(\lambda))/d(\log(\lambda))$ for $N = 50$ and $N = 49$. The two curves rapidly settle down to a value of approximately 1.5 for $\lambda > 50$ but separate noticeably at $\lambda \approx 300$ above which the truncated series cease to be good approximations to $I(\lambda)$. We estimate $d/2 + q$ by taking $\lambda = 290$ where the separation between the two curves is about 0.5×10^{-6}, which is much less than the error $\mathcal{E}$ obtained by approximating the limiting value of $\tilde{I}$ by its value for finite λ. This gives $d/2 + q \approx \tilde{S}_{50}(290) = 1.500017$. The error is obtained by studying how $I(\lambda)/\lambda^{d/2+q}$ settles down to a constant value. For small λ the approach to a constant value is controlled by exponential terms that originate from the suppression of the contribution of the cut, but for larger λ the error is dominated by power corrections to the small ρ behaviour of $\Upsilon(1/\rho)$. A plot of $(S_{50}(\lambda)/\lambda^{1.5} - S_{50}(290)/290^{1.5})\lambda^{2.6}$ reveals oscillations of roughly equal amplitude approximately equal to 12, so that the error in approximating the $\lambda \to \infty$ value of $I(\lambda)/\lambda^{d/2+q}$ is of order $12/\lambda^{2.6}$ leading to an estimate of the error $\mathcal{E} = \pm 2 \times 10^{-5}$. So we conclude that $q = 0$ to the accuracy of our calculations. Having obtained q we can estimate υ from the $\lambda = 290$ value of $S_{50}\Gamma(2.5)/\lambda^{1.5}$ as 0.999998 with an error of $\pm 10^{-5}$. We have repeated the calculation of q for various values of d. The results are shown in figure 2. We have plotted our estimate of q (multiplied by d^3 to make the results for large d visible) for d varying in steps of 0.025 from 0.05 to 10.5. The results are consistent with $q = 0$, which is the

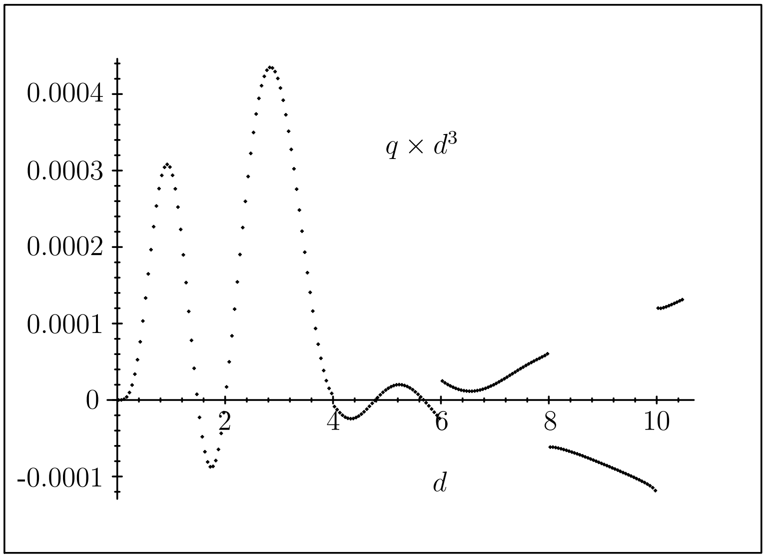

Figure 2: q as a function of d.

exact result of [2]. By calculating a few values of υ we guessed that its dependence on d is given by

$$\upsilon = \frac{-((d-3)/2)!^2 2^{d-3}}{\sin(d\pi/2)(d-2)!^2} \equiv \tilde{\upsilon}(d)\,, \tag{3.4}$$

and we test this by plotting in figure 3 our numerical estimates of υ divided by the r.h.s. of (3.4) for the same range of d as before. From this, and $q = 0$ we conclude that the φ-dependence of the AdS partition function is given by

$$\log Z[\varphi] = \lim_{\tau\to\infty} \frac{1}{2\tau^d} \int d^d x\, \varphi \tilde{\upsilon}(d)\big(-\tau^2\nabla^2\big)^{d/2} \varphi\,. \tag{3.5}$$

This is a *non-local* expression, even when d is an even integer, thanks to the singularity that then appears in $\tilde{\upsilon}$, since

$$\tilde{\upsilon}(d)\big(-\nabla^2\big)^{d/2}\delta^d(x-y) = \frac{\tilde{c}}{|x-y|^{2d}}, \quad \tilde{c} = \frac{(d-1)(d/2)!(d/2-3/2)!^2}{8\pi^{d/2+1}(d-2)!} \tag{3.6}$$

so that

$$\log Z[\varphi] = \frac{\tilde{c}}{2} \int d^d x\, d^d y\, \frac{\varphi(x)\varphi(y)}{|x-y|^{2d}} \tag{3.7}$$

as in [2].

The calculation of a in (3.2) requires the introduction of a regulator into the functional laplacian of (1.9). We will do this with a cut-off on the eigenvalues, k^2 of $-\nabla^2$, restricting them to be less than $1/(\tau^2 s)$ where s is the square of a fixed proper distance. Thus we replace $\delta^d(x)$ in (3.2) by

$$\theta\Big(s\tau^2\nabla^2 + 1\Big)\delta^d(x) = \int_{k^2<1/(\tau^2 s)} \frac{d^d k}{(2\pi)^d} e^{i\mathbf{k}\cdot x}\,, \tag{3.8}$$

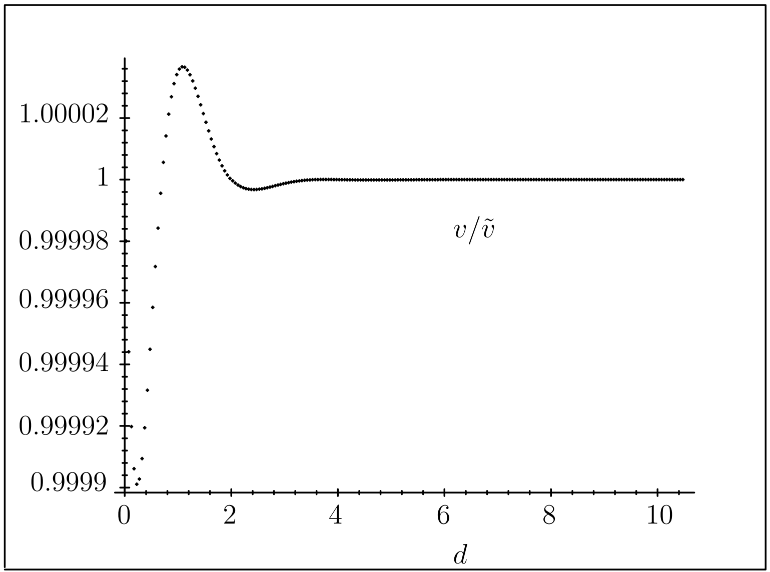

Figure 3: v as a function of d.

where θ is the step-function, giving

$$a(s) = -\frac{\tau^d}{2d}\int_{k^2<1/(\tau^2 s)} d^d k\left(\Gamma\left(\tau^2 k^2\right)+\frac{d}{2}\right) = -\frac{1}{2d}\int_{k^2<1/s} d^d k\left(\Gamma\left(k^2\right)+\frac{d}{2}\right). \quad (3.9)$$

The continuum limit corresponds to taking s to zero. Unfortunately when we substitute our local expansion (2.10) and (3.3) into this we obtain a series that will converge only for large values of s

$$a(s) = -\frac{V_d}{2ds^{d/2}}\sum_{n=0}^{\infty}\frac{b_n}{s^n(2n+d)}\,, \quad (3.10)$$

where V_d/d is the volume of the unit ball in d-dimensions. However our previous arguments imply that $a(s)$ is an analytic function of s in the complex s-plane cut along the negative real axis, so that if $a(s) \sim a_0/s^\nu$ for small s then for large λ

$$\frac{1}{2\pi i}\int\frac{ds}{s}e^{\lambda s}a(s) = -\frac{V_d}{2d}\sum_{n=0}^{\infty}\frac{b_n\lambda^{n+d/2}}{(2n+d)(n+d/2)!} \sim \frac{a_0\lambda^\nu}{\nu!}\,. \quad (3.11)$$

Numerical investigation of this suggests that $\nu = d+1$. The ultraviolet divergence of a can be cancelled by renormalizing the cosmological constant.

The effect of adding a mass term to the action, $V(\phi) = m^2\phi^2$ can be understood quite simply by a change of variables back to the massless action. If we set $\phi = t^{-r}\psi$ in the action

$$S = \frac{1}{2}\int_0^\tau dt\int d^d x\left(t^{1-d}\sum_{\mu=0}^{d}(\partial_\mu\phi)^2 + t^{-d-1}m^2\phi^2\right), \quad (3.12)$$

it becomes

$$S = \frac{1}{2}\int_0^\tau dt \int d^d x \Big(t^{1-d-2r} \sum_{\mu=0}^{d} \big(\partial_\mu \psi\big)^2 + t^{-d-1-2r}\Big(m^2 - r^2 - rd\Big)\psi^2\Big) -$$
$$- \frac{1}{2}\int d^d x \, r\tau^{-d-2r}\psi^2(\tau, x) \tag{3.13}$$

so that if we take $r(r+d) = m^2$ we are left with a boundary term plus the massless action (3.1) with $\Omega = 1/t^{d+2r-1} \equiv 1/t^p$, (The inner product on variations of ψ, from which we can construct the functional integral volume element $\mathcal{D}\psi$, is $\|\delta\psi\|^2 = \int dt\, dx \delta\psi^2/t^{p+2}$.) Consequently if we express $W_{\tau,0}[\tilde{\varphi}, \varphi]$ in terms of ψ it takes the form corresponding to (2.8)

$$\int d^d x \, \frac{\Omega}{\tau}\Big(a + \frac{1}{2}\tilde{\psi}\Big(\Gamma\big(-\tau^2\nabla^2\big) - r\Big)\tilde{\psi} + \tilde{\psi}\Xi\big(-\tau^2\nabla^2\big)\psi + \frac{1}{2}\psi\Upsilon\big(-\tau^2\nabla^2\big)\psi\Big), \tag{3.14}$$

where, as before Γ, Ξ and Υ have the local/short-time expansions (2.10) with solutions (3.3) leading to

$$\log Z[\varphi] = \lim_{\tau\to\infty} \frac{1}{2\tau^{p+1}} \int d^d x \, \psi \tilde{v}(p+1)\Big(\big(-\tau^2\nabla^2\big)^{(p+1)/2} - r\Big)\psi \, . \tag{3.15}$$

A non-trivial limit occurs for positive $p+1$, which means that r is the positive root of $r(r+d) = m^2$, giving

$$\log Z[\varphi] = \frac{c'}{2}\int d^d x \, d^d y \, \frac{\varphi(x)\varphi(y)}{|x-y|^{2d+2r}} \tag{3.16}$$

since $d+p+1 = 2(d+r)$, agreeing with [2]. Taking this limit required wave-function renormalization of φ with $z(\tau) = \tau^r$.

Having obtained $\Psi_{\tau,0}[\tilde{\varphi}, \varphi]$ we can find $\Psi_{\tau,\tau'}[\tilde{\varphi}, \varphi]$ using the self-reproducing property of the Schrödinger functional,

$$\Psi_{\tau,0}[\tilde{\varphi}, \varphi] = \int \mathcal{D}\hat{\varphi}\Psi_{\tau,\tau'}\big[\tilde{\varphi}, \hat{\varphi}\big]\Psi_{\tau',0}\big[\hat{\varphi}, \varphi\big] \, . \tag{3.17}$$

If we denote the logarithm of $\Psi_{\tau,\tau'}[\tilde{\varphi}, \varphi]$ by

$$\int d^d x \Big(a_{\tau,\tau'} + \frac{1}{2}\tilde{\varphi}\Gamma_{\tau,\tau'}\tilde{\varphi} + \tilde{\varphi}\Xi_{\tau,\tau'}\varphi + \frac{1}{2}\varphi\Upsilon_{\tau,\tau'}\varphi\Big) \tag{3.18}$$

then computing the gaussian integral leads to

$$\begin{aligned}
\Upsilon_{\tau,0} &= \Upsilon_{\tau',0} - \big(\Upsilon_{\tau,\tau'} + \Gamma_{\tau',0}\big)^{-1}\Xi^2_{\tau',0} \, , \\
\Xi_{\tau,0} &= -\Xi_{\tau',0}\big(\Upsilon_{\tau,\tau'} + \Gamma_{\tau',0}\big)^{-1}\Xi_{\tau,\tau'} \, , \\
\Gamma_{\tau,0} &= \Gamma_{\tau,\tau'} - \big(\Upsilon_{\tau,\tau'} + \Gamma_{\tau',0}\big)^{-1}\Xi^2_{\tau,\tau'}
\end{aligned} \tag{3.19}$$

hence

$$\begin{aligned}
\Upsilon_{\tau,\tau'} &= \big(\Upsilon_{\tau',0}-\Upsilon_{\tau,0}\big)^{-1}\Xi^2_{\tau',0}-\Gamma_{\tau',0}\,,\\
\Xi_{\tau,\tau'} &= -\big(\Upsilon_{\tau',0}-\Upsilon_{\tau,0}\big)^{-1}\Xi_{\tau,0}\Xi_{\tau',0}\,,\\
\Gamma_{\tau,\tau'} &= \Gamma_{\tau,0}+\big(\Upsilon_{\tau',0}-\Upsilon_{\tau,0}\big)^{-1}\Xi^2_{\tau,0}\,.
\end{aligned} \tag{3.20}$$

So, in terms of Γ, Ξ and Υ

$$\begin{aligned}
\Upsilon_{\tau,\tau'} &= \tau'^{-(p+1)}\Bigg(\bigg(\Upsilon\big(-\tau'^2\nabla^2\big)-\left(\frac{\tau'}{\tau}\right)^{p+1}\Upsilon\big(-\tau^2\nabla^2\big)\bigg)^{-1}\times\\
&\qquad\times\,\Xi\big(-\tau'^2\nabla^2\big)^2-\Gamma\big(-\tau'^2\nabla^2\big)\Bigg),\\
\Xi_{\tau,\tau'} &= \big(\tau\tau'\big)^{-(p+1)}\Big(\tau^{-(p+1)}\Upsilon\big(-\tau^2\nabla^2\big)-\tau'^{-(p+1)}\Upsilon\big(-\tau'^2\nabla^2\big)\Big)^{-1}\times\\
&\qquad\times\,\Xi\big(-\tau^2\nabla^2\big)\Xi\big(-\tau'^2\nabla^2\big),\\
\Gamma_{\tau,\tau'} &= \tau^{-(p+1)}\Bigg(\Gamma\big(-\tau^2\nabla^2\big)+\bigg(\left(\frac{\tau}{\tau'}\right)^{p+1}\Upsilon\big(-\tau'^2\nabla^2\big)-\Upsilon\big(-\tau^2\nabla^2\big)\bigg)^{-1}\times\\
&\qquad\times\,\Xi\big(-\tau^2\nabla^2\big)^2\Bigg).
\end{aligned} \tag{3.21}$$

4. Conformal anomaly for scalar fields

So far we have worked with a boundary sphere in which the curvature is effectively placed at spatial infinity. To discuss the conformal anomaly it will be more convenient to smooth this out, so instead of the metric (1.1) we will now use

$$ds^2=\frac{1}{t^2}\bigg(dt^2+\sum_{i,j}g_{ij}\,dx^i\,dx^j\bigg). \tag{4.1}$$

with $i,j=1,\ldots,d$. We take g_{ij} to be the metric of a d-dimensional sphere of large radius, r, obtained, for example, by stereographic projection onto the plane parametrized by x. The conformal anomaly measures the response of the free energy, which is the field independent part of $\log Z$, to a Weyl transformation of g_{ij}. We wish to compute this as $r\to\infty$ when we recover the AdS metric (1.1). When the boundary is a two-dimensional sphere the free energy should change by $\int d^2x\sqrt{g}Rc\delta\rho/(48\pi)=c\delta\rho/6$ when $\delta g_{ij}=g_{ij}\delta\rho$ where R is the curvature of the boundary and c the central charge of the Virasoro algebra. More generally there will be a conformal anomaly when the boundary has an even number of dimensions, $2N$ say. We will continue to keep d a continuous variable allowing it to tend to $2N$ at the end of our calculations. Now $\log Z[\varphi]=\lim_{\tau\to\infty,\tau'\to 0}W_{\tau,\tau'}[0,\varphi]$ so we need to compute $W_{\tau,\tau'}[0,\varphi]$ in the presence of the curved metric g_{ij}, which we can still do using our previous technique.

Since $r \to \infty$ it will be sufficent to find the free energy from a derivative expansion. In this section we consider the scalar field. We will discuss a free massive theory, because at one-loop the calculation is the same as for an interacting scalar theory. The Schrödinger equation takes the same form as before, (1.9), provided that Ω and Ω' acquire a factor of $\sqrt{\det g}$ and that ∇ is the covariant derivative constructed from g_{ij}, and the solution is again of the form (2.8), but with a no longer constant, but depending on g_{ij} and τ. If we set $g_{ij} = \delta_{ij} + h_{ij}(x)$, and treat h_{ij} as a source in the same way that we treated φ and $\tilde{\varphi}$ as sources, we can generalize our earlier discussion to argue that $W_{\tau,\tau'}[\tilde{\varphi}^\rho, \varphi^\rho, g^\rho_{ij}]$ is analytic in the cut ρ-plane, where $g^\rho_{ij}(x) = g_{ij}(x/\sqrt{\rho})$. Again, this allows us to reconstruct the large τ solution of the Schrödinger equation from the small τ solution for which we have the local expansion (2.10). By using g_{ij} the Schrödinger functional can be made invariant under reparametrizations of the space-like variables, giving

$$W_{\tau,\tau'}\left[\tilde{\varphi}^\rho, \varphi^\rho, g^\rho_{ij}\right] = W_{\tau,\tau'}\left[\tilde{\varphi}, \varphi, \rho g_{ij}\right] = W_{\tau/\sqrt{\rho},\tau'/\sqrt{\rho}}\left[\tilde{\varphi}, \varphi, g_{ij}\right], \tag{4.2}$$

which firstly shows that the functional evaluated for the scaled fields is the same as the Weyl transformed functional, and secondly that this transformation can be absorbed into a rescaling of τ and τ'. This implies that $W_{\tau,\tau'}[\tilde{\varphi}, \varphi]$ is analytic in the complex τ-plane cut along the negative real axis. In particular, since the part that is quadratic in $\tilde{\varphi}$ is $\tau^{-d} \int d^dx \tilde{\varphi} \Gamma(-\tau^2\nabla^2)\tilde{\varphi}$, it follows that $\Gamma(-(\tau^2/\rho)\nabla^2)$ is analytic in the cut ρ-plane. This allows us to express $\Gamma(-\tau^2/\nabla^2)$ for arbitrary τ in terms of the local expansion (2.10)

$$\begin{aligned}\Gamma\big(-\tau^2\nabla^2\big) &= \lim_{\lambda\to\infty} \frac{1}{2\pi i}\int_C \frac{d\rho}{\rho-1} e^{\lambda(\rho-1)}\Gamma\left(-\left(\frac{\tau^2}{\rho}\right)\nabla^2\right) = \\ &= \lim_{\lambda\to\infty}\sum_{n=0}^{\infty} b_n \frac{1}{2\pi i}\int_C \frac{d\rho}{\rho-1}\frac{e^{\lambda(\rho-1)}}{\rho^n}\big(-\tau^2\nabla^2\big)^n\end{aligned} \tag{4.3}$$

since the large contour, C, on which we can use the local expansion, (2.10), can be collapsed to a contribution from the cut, which is suppressed for large positive λ and the pole at $\rho = 1$ which gives us the l.h.s. Expanding the denominators in powers of $1/\rho$ gives, for example

$$\Gamma\big(-\tau^2\nabla^2\big) = \lim_{\lambda\to\infty}\sum_{n,r=0}^{\infty}(-)^r\frac{b_n\lambda^{n+r}(-\tau^2\nabla^2)^n}{(n-1)!r!(n+r)}. \tag{4.4}$$

The free energy is the $\tau \to \infty, \tau' \to 0$ limit of $F[\tau', g_{ij}] = \int d^dx a_{\tau,\tau'}$. A Weyl scaling of g_{ij} can be compensated by scaling τ, and τ' so when $\delta g_{ij} = g_{ij}\delta\rho$ the change in F is

$$\delta F = -\frac{\delta\rho}{2}\left(\tau\frac{\partial F}{\partial \tau} + \tau'\frac{\partial F}{\partial \tau'}\right). \tag{4.5}$$

F satisfies equations similar to regulated versions of (3.2) (even if we include interactions), that follow from the Schrödinger eqs. (1.9) and (1.11). If we use the same regulator as before, cutting off the large eigenvalues of ∇^2, then

$$\frac{\partial F}{\partial \tau} = \frac{1}{2\tau} \int d^d x \left(\tau^{p+1} \Gamma_{\tau,\tau'} + \frac{p+1}{2} \right) \theta\big(s\tau^2 \nabla^2 + 1 \big) \delta^d(x-y) \,|_{x=y} \; . \tag{4.6}$$

and

$$-\frac{\partial F}{\partial \tau'} = \frac{1}{2\tau'} \int d^d x \left(\tau'^{p+1} \Upsilon_{\tau,\tau'} - \frac{p+1}{2} \right) \theta\big(s\tau'^2 \nabla^2 + 1 \big) \delta^d(x-y) \,|_{x=y} \; . \tag{4.7}$$

If we represent the step function by

$$\theta(x) = \frac{1}{2\pi i} \int_{C'} \frac{dy}{y} e^{iyx} \,, \tag{4.8}$$

with C' a contour running just below the real axis, and if f is some function of $-\nabla^2$ then

$$f\left(-\nabla^2\right) \theta\big(s\tau^2 \nabla^2 + 1 \big) \delta^d(x-y) = \frac{1}{2\pi i} \int_{C'} \frac{dy}{y} e^{iy} f\left(\frac{i}{s\tau^2} \frac{\partial}{\partial y} \right) e^{iys\tau^2 \nabla^2} \delta^d(x-y) \,. \tag{4.9}$$

Now $e^{it\nabla^2} \delta^d(x-y) \equiv H(t,x,y)$ satisfies the finite-dimensional Schrödinger equation

$$i \frac{\partial}{\partial z} H = -\nabla^2 H \,, \qquad H(0,x,y) = \delta^d(x-y) \,. \tag{4.10}$$

At coincident argument H has the small z expansion in powers of derivatives of g_{ij}

$$H(z,x,x) \sim \frac{\sqrt{g}}{(4\pi i z)^{d/2}} \sum_{n=0}^{\infty} a_n(x) z^n \,. \tag{4.11}$$

The $a_n(x)$ are scalars made out of the metric and its derivatives at x, and $a_0(x) = 1$. They depend on the radius of the sphere as $a_n \sim r^{-2n}$. Thus we can express (4.9) as

$$\sum_{n=0}^{\infty} \frac{1}{(4\pi i)^{d/2}} \left(\int \frac{d^d x}{\tau^d} \sqrt{g} a_n(x) \tau^{2n} \right) \frac{1}{2\pi i} \int_{C'} \frac{dy}{y} e^{iy} f\left(\frac{i}{s\tau^2} \frac{\partial}{\partial y} \right) (sy)^{n-d/2} . \tag{4.12}$$

Only a finite number of the a_m survive as $r \to \infty$. For the case of a two-dimensional boundary these are just $a_0 = 1$ and $a_1 = iR/6$, and the conformal anomaly is proportional to a_1. When the boundary has $2N$ dimensions the conformal anomaly is proportional to a_N. If we assume that f has a series expansion, $f(x) = \sum \tilde{f}_n x^n$ then we can compute the relevant integral in (4.12) as

$$\begin{aligned} &\frac{1}{2\pi i} \int_{C'} \frac{dy}{y} e^{iy} f\left(\frac{i}{s\tau^2} \frac{\partial}{\partial y} \right) (sy)^{N-d/2} = \\ &\quad = \sum \tilde{f}_n s^{N-d/2} \left(-\frac{i}{s\tau^2} \right)^n \frac{\sin(\pi d/2)(-1)^{N+1}(N-d/2)!}{\pi(N-d/2-n)} = \\ &\quad = \sin(\pi d/2)(-1)^N (N-d/2)! \frac{1}{\pi} \int_s^{\infty} ds' s'^{N-1-d/2} f\left(-\frac{i}{s'\tau^2} \right), \end{aligned} \tag{4.13}$$

where we have taken $N < d/2$. If f has a finite limit, f_{lim}, as $d \downarrow 2N$ then, for d close to $2N$ this becomes

$$\left(\frac{d}{2-N}\right)\int_s^\infty ds' {s'}^{N-1-d/2} f\left(-\frac{i}{s'\tau^2}\right) =$$
$$= s^{N-d/2} f\left(-\frac{i}{s\tau^2}\right) + \int_s^\infty ds' {s'}^{N-d/2}\frac{d}{ds'} f\left(-\frac{i}{s'\tau^2}\right) \tag{4.14}$$

which tends to $f_{\text{lim}}(0)$ as $d \downarrow 2N$. Putting all this together we obtain the conformal anomaly as the large τ small τ' limit of

$$\delta F = -\frac{\delta\rho}{4}\left(p+1+\hat{\Gamma}-\hat{\Upsilon}\right)\frac{1}{(4\pi i)^N}\int d^d x\sqrt{g}a_N(x)\,, \tag{4.15}$$

where

$$\hat{\Gamma} = \lim_{\xi\to 0}\left(\lim_{d\downarrow 2N}\left(\Gamma\big(-\tau^2\xi\big) + \left\{\left(\frac{\tau}{\tau'}\right)^{p+1}\Upsilon\big(-{\tau'}^2\xi\big) - \Upsilon\big(-\tau^2\xi\big)\right\}^{-1}\Xi\big(-\tau^2\xi\big)^2\right)\right),$$
$$\hat{\Upsilon} = \lim_{\xi\to 0}\left(\lim_{d\downarrow 2N}\left(\left\{\Upsilon\big(-{\tau'}^2\xi\big) - \left(\frac{\tau'}{\tau}\right)^{p+1}\Upsilon\big(-\tau^2\xi\big)\right\}^{-1}\Xi\big(-{\tau'}^2\xi\big)^2 - \Gamma\big(-{\tau'}^2\xi\big)\right)\right). \tag{4.16}$$

From the series expansions (3.3) we see that $\hat{\Gamma} = b_0 = -(p+1)$ and for generic values of p we have $\hat{\Upsilon} = 0$. The order of the limits is important since when p approaches an odd integer as $d \downarrow 2N$ the coefficient f_N diverges so that for $\xi \neq 0$ there is a suppression of $\{\Upsilon(-{\tau'}^2\xi) - (\tau'/\tau)^{p+1}\Upsilon(-\tau^2\xi)\}^{-1}$ thus $\hat{\Upsilon} = -\hat{\Gamma}$ which is just $-b_0$. Since $p+1 = \sqrt{d^2+4m^2} = 2\sqrt{N^2+m^2}$ we have that when $\sqrt{N^2+m^2}$ is an integer, $\tilde{N}$, the conformal anomaly is

$$\delta F = \frac{\delta\rho}{2}\frac{\tilde{N}}{(4\pi i)^N}\int d^{2N}x\sqrt{g}a_N(x) \tag{4.17}$$

otherwise it vanishes. In particular for $\mathbf{d} = 2$ we have that the central charge of the Virasoro algebra, c, equals $\tilde{N}$ when $m = \sqrt{\tilde{N}^2 - 1}$ and vanishes otherwise. For $\mathbf{d} = \mathbf{4}$

$$\delta F = -\frac{\delta\rho}{32\pi^2}\int d^4x\sqrt{g}a_2(\mathbf{x})\tilde{N} \tag{4.18}$$

or zero, where

$$a_2 = \frac{1}{180}\left(R_{ijkl}R^{ijkl} - R_{ij}R^{ij} - 6\Box R + \frac{5}{2}R^2\right). \tag{4.19}$$

Our conventions for the curvature tensors are as in [12], i.e., $R^i_{jkl} = \partial_l\Gamma^i_{jk} - \cdots$, $R_{ij} = R^k_{ikj}$ and $\Box = g^{ij}\nabla_i\nabla_j$. The term in $\Box R$ can be removed by a counter-term proportional to R^2. The mass condition $m^2 = \tilde{N}^2 - N^2$, corresponds to the mass spectrum of the scalar fields of Supergravity compactified on $AdS_5 \times S^5$, [19], for $d = 2N = 4$, and on $AdS_7 \times S^4$ for $d = 2N = 6$, [20]. Note that for $D = 2N \geq 4$ there are negative values of m^2 with non-vanishing conformal anomaly.

5. Example: free fermion field

Another example is given by a free fermion theory. In order to describe this, it will be necessary to discuss some subtleties arising from the representation of fermion fields.

Wave functionals will be taken to be overlaps $\langle u, u^\dagger|\Psi\rangle$ with a field state $\langle u, u^\dagger|$, upon which fermion operators act as follows. We diagonalize half of the components of the fermion fields

$$\big\langle u, u^\dagger\big|Q_+\hat{\psi} = \sqrt{2}Q_+u\big\langle u, u^\dagger\big|\,, \qquad \big\langle u, u^\dagger\big|\hat{\psi}^\dagger Q_- = \sqrt{2}u^\dagger Q_-\big\langle u, u^\dagger\big|, \tag{5.1}$$

where $Q_\pm = (1/2)(1 \pm Q)$ are for the moment arbitrary projection operators. The other half are represented by functional differentiation

$$\big\langle u, u^\dagger\big|Q_-\hat{\psi} = \frac{1}{\sqrt{2}}Q_-\frac{\delta}{\delta u^\dagger}\big\langle u, u^\dagger\big|\,, \qquad \big\langle u, u^\dagger\big|\hat{\psi}^\dagger Q_+ = \frac{1}{\sqrt{2}}\frac{\delta}{\delta u}Q_+\big\langle u, u^\dagger\big|\,. \tag{5.2}$$

We can make the dependence of the field state on the Grassmann-valued source fields explicit by writing

$$\big\langle u, u^\dagger\big| = \big\langle Q\big| \exp\sqrt{2}\,\mathrm{tr}\int d^dx\Big(u^\dagger Q_-\hat{\psi} - \hat{\psi}^\dagger Q_+ u\Big)\,, \tag{5.3}$$

where $\langle Q|$ is defined by

$$\big\langle Q\big|Q_+\hat{\psi} = \big\langle Q\big|\hat{\psi}^\dagger Q_- = 0\,. \tag{5.4}$$

Thus the choice of Q corresponds to a choice of Dirac sea, though *not* a physical Dirac sea; these constraints are merely artifacts of our choice of representation. On the other hand this choice should not be considered completely arbitrary; since the field-states parametrized by $\langle Q|$ are nonphysical, we must be careful that their overlaps with physical states are well defined and nonvanishing. Also, we may wish to include gauge interactions; if we want wave-functionals to be invariant under local gauge transformations, we find that we must choose Q to be a local, field independent operator. In particular, if we choose $Q_\pm$ to be projectors onto +ve/-ve energy eigenstates, the resulting wave-functionals do *not* satisfy Gauss' law [13, 14]. For reasons that will become clear, in AdS space we will choose $Q = \pm\gamma^0$.

Now define

$$\big\langle\big\langle u, u^\dagger\big| \equiv \big\langle u, u^\dagger\big|e^{\mathrm{tr}\int d^dx[u^\dagger Q u]} = \big\langle Q\big|e^{\sqrt{2}\,\mathrm{tr}\int d^dx(u^\dagger\hat{\psi} - \hat{\psi}^\dagger u)}\,, \tag{5.5}$$

using which it may be verified that our representation coincides with that given in [13].

$$\begin{aligned}\big\langle\big\langle u, u^\dagger\big|\hat{\psi} &= \frac{1}{\sqrt{2}}\left(u + \frac{\delta}{\delta u^\dagger}\right)\big\langle\big\langle u, u^\dagger\big|\,,\\ \big\langle\big\langle u, u^\dagger\big|\hat{\psi}^\dagger &= \frac{1}{\sqrt{2}}\left(u^\dagger + \frac{\delta}{\delta u}\right)\big\langle\big\langle u, u^\dagger\big|\,.\end{aligned} \tag{5.6}$$

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Similarly, if we define

$$\big|v, v^\dagger\big\rangle\!\big\rangle = e^{\sqrt{2}\,\mathrm{tr}\int d^dx(v^\dagger\hat{\psi}-\hat{\psi}^\dagger v)}\big|Q\big\rangle\,, \tag{5.7}$$

with $Q_-\hat{\psi}|Q\rangle = \hat{\psi}^\dagger Q_+|Q\rangle = 0$, then we have

$$\begin{aligned}\hat{\psi}\big|v, v^\dagger\big\rangle\!\big\rangle &= \frac{1}{\sqrt{2}}\left(v - \frac{\delta}{\delta v^\dagger}\right)\big|v, v^\dagger\big\rangle\!\big\rangle\,,\\ \hat{\psi}^\dagger\big|v, v^\dagger\big\rangle\!\big\rangle &= \frac{1}{\sqrt{2}}\left(v^\dagger - \frac{\delta}{\delta v}\right)\big|v, v^\dagger\big\rangle\!\big\rangle\,.\end{aligned} \tag{5.8}$$

Written in the form (5.6) or (5.8), the representation is reducible, but we are free to take

$$Q_-u = u^\dagger Q_+ = 0\,, \tag{5.9}$$

and

$$Q_+v = v^\dagger Q_- = 0\,, \tag{5.10}$$

which removes the reducibility. To avoid taking functional derivatives with respect to constrained fields, however, we find it most convenient to work with the unconstrained sources, and impose the constraints at the end of our calculations. The states $|v, v^\dagger\rangle$ and $\langle u, u^\dagger|$ depend only on the unconstrained components in any case, and the relative factor of $e^{\mathrm{tr}\int d^dx[u^\dagger Qu]}$ appearing in $\langle\!\langle u, u^\dagger|$ merely ensures that we stay in the appropriate Fock space when we apply field operators in accordance with (5.6) and (5.8).

Imposing the constraints will nevertheless be of physical importance; for example in the AdS /CFT correspondence they cause a Dirac fermion in the AdS theory to become a chiral fermion in the boundary theory. This is in agreement with the findings of other authors, for example in [15].

Now we are interested in the Schrödinger functional

$$\Psi_{\tau,\tau'}\Big[u, u^\dagger, v, v^\dagger\Big] = \big\langle\!\big\langle u, u^\dagger\big|T\exp\left(-\int_{\tau'}^{\tau} dt\,\hat{H}(t)\right)\big|v, v^\dagger\big\rangle\!\big\rangle\,, \tag{5.11}$$

so we need to find the overlap $\langle\!\langle u, u^\dagger|v, v^\dagger\rangle\!\rangle$ which is the coincident time limit of the above. The definitions (5.4) and (5.5) give rise to the following equations

$$\begin{aligned}0 &= \left(\hat{\psi}^\dagger - \sqrt{2}u^\dagger\right)Q_-\big\langle\!\big\langle u, u^\dagger\big|v, v^\dagger\big\rangle\!\big\rangle = Q_-\left(\hat{\psi} + \sqrt{2}v\right)\big\langle\!\big\langle u, u^\dagger\big|v, v^\dagger\big\rangle\!\big\rangle =\\ &= \left(\hat{\psi}^\dagger + \sqrt{2}v^\dagger\right)Q_+\big\langle\!\big\langle u, u^\dagger\big|v, v^\dagger\big\rangle\!\big\rangle = Q_+\left(\hat{\psi} - \sqrt{2}u\right)\big\langle\!\big\langle u, u^\dagger\big|v, v^\dagger\big\rangle\!\big\rangle\,,\end{aligned} \tag{5.12}$$

where we have used the canonical commutation relations. The field operators in (5.12) may be represented by either pair of source fields, and the equations solved to give

$$\langle\!\langle u, u^\dagger | v, v^\dagger \rangle\!\rangle = \exp \operatorname{tr} \int d^d x \Big(u^\dagger Q u - u^\dagger 2 Q_- v + v^\dagger 2 Q_+ u + v^\dagger Q v \Big) . \tag{5.13}$$

We are now ready to consider the AdS metric (1.1). With the choice of vielbein

$$e^a_\mu = t^{-1} \delta^a_\mu , \tag{5.14}$$

the euclidean action is given by[1]

$$S = \int d^{d+1}x \sqrt{g}\, \bar\psi \big(t\gamma \cdot D - m \big) \psi = \int d^{d+1}x t^{-d} \bar\psi \Big(\gamma \cdot \partial - \frac{\gamma^0 d}{2t} - \frac{m}{t} \Big) \psi , \tag{5.15}$$

since according to (1.1), $\sqrt{g} = t^{-d-1}$, and the spin covariant derivative is $D_\mu = \partial_\mu - (1/t)\Sigma_{0\mu}$. Changing variables to $\phi = t^{-d/2}\psi$ and $\phi^\dagger = t^{-d/2}\psi^\dagger$ the action becomes

$$S = \int d^{d+1}x \bar\phi \Big(\gamma \cdot \partial - \frac{m}{t} \Big) \phi , \tag{5.16}$$

and for $m = 0$ it coincides with the flat-space action. As in the bosonic case, if we also put $\bar t = \ln t$, the volume element in the corresponding path-integral becomes the usual flat-space one induced by $\|\delta\phi\|^2 = \int d\bar t\, dx\, \delta\phi^\dagger\, \delta\phi$. Thus we can make use of the representation (5.6) for ϕ and $\phi^\dagger$. The integrands in (5.3) and (5.13) do not aquire a factor from the metric, as this has been absorbed into the definition of the fields.

The partition function is again given by the $\tau' \to 0$, $\tau \to \infty$ limit of the Schrödinger functional, with $u = u^\dagger = \lim_{|\mathbf{x}|\to\infty} v(\mathbf{x}) = \lim_{|\mathbf{x}|\to\infty} v^\dagger(\mathbf{x}) = 0$. In path-integral form the Schrödinger functional is

$$\Psi_{\tau,\tau'} \Big[u, u^\dagger, v, v^\dagger \Big] = \int \mathcal{D}\phi \mathcal{D}\phi^\dagger e^{-S - S_B} , \tag{5.17}$$

where the boundary term is

$$\begin{aligned} S_B = \int_{x^0=\tau'} & d^d x \Big(\phi^\dagger Q_- \phi - \sqrt{2}\phi^\dagger Q_- v + \sqrt{2} v^\dagger Q_+ \phi \Big) - \\ & - \int_{x^0=\tau} d^d x \Big(\phi^\dagger Q_+ \phi - \sqrt{2}\phi^\dagger Q_+ u + \sqrt{2} u^\dagger Q_- \phi \Big). \end{aligned} \tag{5.18}$$

If ϕ and $\phi^\dagger$ are integrated over freely then we can shift them by solutions to the classical equations of motion. Choosing these solutions to satisfy the boundary conditions corresponding to (5.1)

$$\begin{aligned} t = \tau' : &\quad Q_- \phi = -\sqrt{2} Q_- v , \quad \phi^\dagger Q_+ = -\sqrt{2} v^\dagger Q_+ , \\ t = \tau : &\quad Q_+ \phi = \sqrt{2} u , \quad \phi^\dagger Q_- = \sqrt{2} u^\dagger Q_- \end{aligned} \tag{5.19}$$

[1] Gamma matrices obey $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ throughout this section.

causes the action to separate into a piece depending only on the integration variables and a piece depending only on the classical solution. Our boundary term S_B is thus determined by the conditions (5.1). Note that the classical action does *not* vanish, and there is therefore no need to add any additional boundary term with undetermined coefficients, as in [15]. (Other authors have discussed boundary terms for fermions [16, 17]).

The Schrödinger equation is

$$-\frac{\partial}{\partial\tau}\Psi_{\tau,\tau'} = \hat{H}\Psi_{\tau,\tau'}\,, \tag{5.20}$$

where

$$\hat{H} = \frac{1}{2}\int d^dx\left(u^\dagger + \frac{\delta}{\delta u}\right)h\left(u + \frac{\delta}{\delta u^\dagger}\right), \quad h = \left(\gamma^0\gamma^i\partial_i - \frac{\gamma^0 m}{\tau}\right). \tag{5.21}$$

Now the logarithm of the partition function is obtained as $\lim_{\tau\to\infty} W_{\tau,0}[0,0,v,v^\dagger]$, where $W_{\tau,0}[u,u^\dagger,v,v^\dagger] = \log\Psi_{\tau,0}[u,u^\dagger,v,v^\dagger]$ may be expanded in analogy with (2.8) as

$$\int d^dx\Big\{f + u^\dagger\Gamma\Big(\tau\gamma_0\gamma^i\partial_i\Big)u + u^\dagger\Xi\Big(\tau\gamma_0\gamma^i\partial_i\Big)v + v^\dagger\Pi\Big(\tau\gamma_0\gamma^i\partial_i\Big)u + v^\dagger\Upsilon\Big(\tau\gamma_0\gamma^i\partial_i\Big)v\Big\}\,. \tag{5.22}$$

Substituting (5.22) into (5.20) gives

$$\begin{gathered}
\dot{\Gamma} = -\frac{1}{2}(1-\Gamma)h(1+\Gamma)\,, \qquad \dot{\Xi} = -\frac{1}{2}(1-\Gamma)h\Xi\,, \qquad \dot{\Pi} = \frac{1}{2}\Pi h(1+\Gamma)\,, \\
\dot{\Upsilon} = \frac{1}{2}\Pi h\Xi\,, \qquad \dot{f} = \frac{1}{2}\,\mathrm{Tr}\,h(1+\Gamma)\delta^d(x)|_{\mathbf{x}=0}\,, \\
\Gamma(0) = \Upsilon(0) = Q\,, \qquad \Xi(0) \sim -2Q_-\,, \qquad \Pi(0) \sim 2Q_+
\end{gathered} \tag{5.23}$$

where the initial conditions are read off from (5.13). We will find that $\Xi(0)$ and $\Pi(0)$ diverge as a result of the ill-defined nature of $\lim_{\tau\to 0}\int_0^\tau (m/t)dt$, but that $\lim_{\tau\to\tau'}\Psi_{\tau,\tau'}$ is well defined and given by (5.13) for all $\tau' \neq 0$.

As may be verified by direct substitution, the equation for Γ is solved by

$$\Gamma = (\Sigma - Q_-)(\Sigma + Q_-)^{-1}, \tag{5.24}$$

with Σ satisfying the Dirac equation $\dot{\Sigma} + h\Sigma = 0$. Making the ansatz

$$\Sigma = \sum_{n=0}^{\infty} a_n\Big(-\tau\gamma^0\gamma^i\partial_i\Big)^n Q_+\tau^{-m}\,, \tag{5.25}$$

leads (for the specific choice $Q = \gamma^0$) to the recurrence relation $a_n = a_{n-1}/(n + m(1-(-1)^n))$. The boundary condition is satisfied if $a_0 = 1$ and $m \geq 0$, and we can explicitly sum the series in terms of Bessel functions. In momentum space

$$\Sigma = \Gamma\left(\frac{1}{2}+m\right)\left(\frac{p\tau}{2}\right)^{1/2-m}\Big(I_{m-1/2}(p\tau)+PI_{m+1/2}(p\tau)\Big)Q_+\tau^m = \\ = (E+O)Q_+\,. \tag{5.26}$$

Here $P=(i\gamma^0\gamma\cdot\mathbf{p})/|\mathbf{p}|$; the operators $(1/2)(1\pm P)$ project onto +ve/-ve eigenvalues of the massless flat-space hamiltonian $\gamma^0\gamma^i\partial_i$. Substituting back into (5.24) we find that

$$\Gamma = Q+2Q_-OE^{-1} = Q+2Q_-P\frac{I_{m+1/2}(p\tau)}{I_{m-1/2}(p\tau)}\,. \tag{5.27}$$

Next, as may again be verified by substitution, the equation for Π has the solution

$$\Pi = 2Q_+(\Sigma+Q_-)^{-1} = 2Q_+E^{-1}\,, \tag{5.28}$$

which gives the expected divergent behaviour at $\tau=0$.

Now consider the equation for Ξ. We can rewrite Γ in the following way: $\Gamma = -(\bar\Sigma+Q_+)^{-1}(\bar\Sigma-Q_+)$ where

$$\bar\Sigma = Q_+(E-O)\,. \tag{5.29}$$

which satisfies $\dot{\bar\Sigma}-\bar\Sigma h=0$. This enables us to find the solution

$$\Xi = -2\Big(\bar\Sigma+Q_-\Big)^{-1}Q_- = -2Q_-E^{-1}\,. \tag{5.30}$$

Finally, to solve the equation for Υ we put $\Upsilon = \Pi R\Xi + Q$ where R satisfies $\{R,Q\}=0$. Substituting into (5) gives

$$\Pi\Big(2\dot R - \Big(1+4ROE^{-1}\Big)\gamma^0\gamma^i\partial_i\Big)\Xi = 0\,. \tag{5.31}$$

Now define $\tilde O$, $\tilde E$ such that $O\to\tilde O$, $E\to\tilde E$ as $m\to -m$. The correponding expansion coefficients $\tilde a_n$ are divergent as $m-1/2$ approaches an integer, so we keep m variable, allowing it to approach such values within convergent expressions. This is the analogue of keeping d variable in the scalar case. The necessity for such regularisation is due to our working in momentum space, rather than configuration space. Using the identity $E\tilde E - O\tilde O = 1$, we find that $R=-(1/2)\tilde O E$, so that

$$\Upsilon = Q+2Q_+\tilde O E^{-1} = Q+2CQ_+Pp^{2m}\frac{I_{1/2-m}(p\tau)}{I_{m-1/2}(p\tau)}\,, \tag{5.32}$$

where $C=2^{-2m}(-1/2-m)!/(m-1/2)!$. This is nonlocal even when $m=n+1/2$ for some integer n causing the Bessel functions to cancel because then C diverges and we have a similar situation to that which we encountered in the bosonic case.

The large τ behaviour of Υ is easily found

$$\lim_{\tau\to\infty}\Upsilon = 2CQ_+Pp^{2m}\,. \tag{5.33}$$

Fourier transforming, we find that the partition function is given by

$$\log Z\Big[v, v^\dagger\Big] = \int d^dx\, d^dy\, v^\dagger(\mathbf{x})\left(K\gamma^0 \frac{\gamma\cdot(x-y)}{|x-y|^{d+2m+1}}\right)v(y)\,, \tag{5.34}$$

where $K = -2((d-1)/2+m)!/((m-1/2)!\pi^{d/2})$. Here we have imposed the constraints (5.9). This is the correct two-point function for a quasi-primary fermion field of scaling dimension $d/2+m$.

Now all this assumed that $m \geq 0$; but if $m < 0$ the above argument holds with m replaced by $-m$ provided we take $Q = -\gamma^0$. Thus we conclude that the scaling dimension is $d/2+|m|$ in general.

An incidental point of interest is the relationship of our solution of the Schrödinger equation to the classical field configuration ϕ to which it corresponds. This is not quite the same as the solution Σ to the Dirac equation which we found, because the latter is divergent as $\tau\to\infty$. However, defining $\tilde\Sigma$ by $\Sigma\to\tilde\Sigma$ as $m\to -m$, we find that $P\tilde\Sigma$ also solves the Dirac equation. Taking a suitable linear combination of these two solutions (and allowing them to operate on the boundary value v) we can construct a unique field configuration which satisfies the appropriate boundary conditions and is finite as $\tau\to\infty$. By Fourier transforming this configuration, we found that it coincides exactly with that given in [15] (when we change back to the variables in the original action).

Now that we have found $\Psi_{\tau,0}$, we can construct $\Psi_{\tau,\tau'}$ from the self-reproducing property. The inner-product of wave-functionals follows from (5.13)

$$\begin{aligned}\langle 1|2\rangle &= \int DuDu^\dagger DvDv^\dagger \langle 1\big|u,u^\dagger\big\rangle\big\langle u,u^\dagger\big|v,v^\dagger\big\rangle\big\langle v,v^\dagger\big|2\big\rangle = \\ &= \int DuDu^\dagger DvDv^\dagger \langle 1\big|u,u^\dagger\big\rangle\big\langle v,v^\dagger\big|2\big\rangle e^{2v^\dagger u - 2u^\dagger v}\,.\end{aligned} \tag{5.35}$$

It is important to note that we integrate over the *constrained* fields, (5.9), which reflect the true functional dependence of the wave-functionals. This allows us to drop the Q-dependence from Γ, etc., so we write the logarithm of $\Psi_{\tau,\tau'}$ as

$$\int d^dx\Big\{f_{\tau,\tau'} + u^\dagger\Gamma_{\tau,\tau'}u + u^\dagger\Xi_{\tau,\tau'}v - v^\dagger\Xi_{\tau,\tau'}u + v^\dagger\Upsilon_{\tau,\tau'}v\Big\}, \tag{5.36}$$

and we have $\Gamma_{\tau,0} = 2OE^{-1}$, $\Xi_{\tau,0} = -2E^{-1}$, and $\Upsilon_{\tau,0} = 2\tilde{O}E^{-1}$. Note that $\Pi_{\tau,0} = -\Xi_{\tau,0}$. Then from

$$\Psi_{\tau,0}\Big[\tilde u,\tilde u^\dagger,\tilde v,\tilde v^\dagger\Big] = \int d^dx DuDu^\dagger DvDv^\dagger \Psi_{\tau,\tau'}\Big[\tilde u,\tilde u^\dagger,u,u^\dagger\Big]\Psi_{\tau',0}\Big[v,v^\dagger,\tilde v,\tilde v^\dagger\Big]e^{2v^\dagger u-2u^\dagger v}\,, \tag{5.37}$$

we obtain

$$
\begin{aligned}
\Gamma_{\tau,0} &= \Gamma_{\tau,\tau'} + \Xi^2_{\tau,\tau'}\Upsilon^{-1}_{\tau,\tau'} - 4\Xi^2_{\tau,\tau'}\Upsilon^{-2}_{\tau,\tau'}\Big(4\Upsilon^{-1}_{\tau,\tau'} + \Gamma_{\tau',0}\Big)^{-1}, \\
\Xi_{\tau,0} &= -2\Xi_{\tau,\tau'}\Xi_{\tau',0}\Upsilon^{-1}_{\tau,\tau'}\Big(4\Upsilon^{-1}_{\tau,\tau'} + \Gamma_{\tau',0}\Big)^{-1}, \\
\Upsilon_{\tau,0} &= \Upsilon_{\tau',0} + \Xi^2_{\tau',0}\Big(4\Upsilon^{-1}_{\tau,\tau'} + \Gamma_{\tau,0}\Big)^{-1}
\end{aligned} \tag{5.38}
$$

and hence

$$
\begin{aligned}
\Upsilon_{\tau,\tau'} &= -4\Big(\Xi^2_{\tau',0}\big(\Upsilon_{\tau',0} - \Upsilon_{\tau,0}\big)^{-1} + \Gamma_{\tau',0}\Big)^{-1}, \\
\Xi_{\tau,\tau'} &= \frac{1}{2}\Xi_{\tau',0}\Pi_{\tau,0}\big(\Upsilon_{\tau',0} - \Upsilon_{\tau,0}\big)^{-1}\Upsilon_{\tau,\tau'}\,, \\
\Gamma_{\tau,\tau'} &= \Gamma_{\tau,0} - \frac{1}{4}\Xi^2_{\tau',0}\Xi^2_{\tau,0}\big(\Upsilon_{\tau',0} - \Upsilon_{\tau,0}\big)^{-2}\Upsilon_{\tau,\tau'} - \big(\Upsilon_{\tau',0} - \Upsilon_{\tau,0}\big)^{-1}\Xi^2_{\tau,0}\,.
\end{aligned} \tag{5.39}
$$

From this we can check that as $\tau \to \tau' \neq 0$ the Schrödinger functional $\Psi_{\tau,\tau'}$ reduces to $\langle u, u^\dagger | v, v^\dagger \rangle$ as it should.

We can calculate the conformal anomaly in the same way as before. Working with the metric (4.1) the Schrödinger equation is unchanged, except that derivatives become covariant with respect to g and the hamiltonian density aquires a factor of $\sqrt{g}$. We need to introduce a UV regulator, which we would like to write in terms of a heat-kernel expansion, so it is convenient to re-express everything in terms of positive definite operators. Hence, for example, we rewrite the solution (5.27) as

$$
\Gamma\Big(\tau\gamma^0\gamma^i\nabla_i\Big) = Q + 2Q_-\tau\gamma^0\gamma^i\nabla_i\sum_{n=0}^{\infty} d_n\Big(\tau^2 D^2\Big)^n, \tag{5.40}
$$

for some appropriate coefficients d_n . We have defined $D \equiv \gamma^0\gamma^i\nabla_i$.

As before, a Weyl transformation may be implemented by scaling τ, so when $\delta g_{ij} = g_{ij}\delta\rho$ the free energy changes by

$$
\delta F = -\frac{\delta\rho}{2}\int d^d x\left(\tau\frac{\partial f_{\tau,\tau'}}{\partial\tau} + \tau'\frac{\partial f_{\tau,\tau'}}{\partial\tau'}\right). \tag{5.41}
$$

The Schrödinger equations yield equations for F corresponding to regulated versions of (5)

$$
\frac{\partial f}{\partial\tau} = \frac{1}{2}\,\mathrm{tr}\,\Big(h\big(1 + Q + Q_-\Gamma_{\tau,\tau'}\big)\theta\big(1 - s\tau^2 D^2\big)\delta^d(\mathbf{x} - y)\,|_{\mathbf{x}=y}\Big), \tag{5.42}
$$

and

$$
-\frac{\partial f}{\partial\tau'} = \frac{1}{2}\,\mathrm{tr}\,\Big(h\big(1 + Q + Q_+\Upsilon_{\tau,\tau'}\big)\theta\big(1 - s\tau'^2 D^2\big)\delta^d(\mathbf{x} - y)\,|_{\mathbf{x}=y}\Big). \tag{5.43}
$$

Representing the step function by (4.8), we have

$$\theta\Big(1-s\tau^2D^2\Big)\delta^d(\mathbf{x}-y)1=\frac{1}{2\pi i}\int_{C'}\frac{dy}{y}e^{iy}e^{-iys\tau^2D^2}\delta^d(\mathbf{x}-y)1\,, \tag{5.44}$$

where $e^{-izD^2}\delta^d(\mathbf{x}-\mathbf{y})1\equiv H(z,\mathbf{x},\mathbf{y})$ satisfies

$$i\frac{\partial}{\partial z}H=D^2H,\qquad H(0,\mathbf{x},y)=\delta^d(\mathbf{x}-y)1\,, \tag{5.45}$$

and has the small z expansion

$$H(z,\mathbf{x},\mathbf{x})\sim\frac{\sqrt{g}}{(4\pi iz)^{d/2}}\sum_{n=0}^{\infty}\tilde{a}_n(\mathbf{x})z^n\,. \tag{5.46}$$

Thus a general function $g(D^2)$ satisfies

$$\begin{aligned}
&\mathrm{tr}\,\Big(g\big(D^2\big)\theta\Big(1-s\tau^2D^2\Big)\delta^d(\mathbf{x}-y)\Big)=\\
&\quad=\frac{1}{2\pi i}\int_{C'}\frac{dy}{y}e^{iy}\,\mathrm{tr}\left(g\left(\frac{i}{s\tau^2}\frac{\partial}{\partial y}\right)e^{iys\tau^2D^2}\delta^d(x-y)\right)=\\
&\quad=\mathrm{tr}\left(\sum_{n=0}^{\infty}\frac{1}{(4\pi i)^{d/2}}\left(\int\frac{d^dx}{\tau^d}\sqrt{g}\tilde{a}_n(x)\tau^{2n}\right)\frac{1}{2\pi i}\int_{C'}\frac{dy}{y}e^{iy}g\left(\frac{i}{s\tau^2}\frac{\partial}{\partial y}\right)\right)(sy)^{n-d/2},
\end{aligned} \tag{5.47}$$

where only terms up to $n=N$ contribute when $d=2N$. The conformal anomaly is again proportional to $\tilde{a}_N$; in the limit $\tau\to\infty$, $\tau'\to0$ the other terms arising from (5.42) and (5.43) either vanish or reproduce ultraviolet divergences associated with the renormalization of the cosmological constant, etc. Using the same argument as in the scalar case, the term proportional to $\tilde{a}_N$ is

$$\lim_{\xi\to0}\left(g(\xi)\frac{1}{(4\pi i)^N}\int d^dx\sqrt{g}\tilde{a}_n(x)\right). \tag{5.48}$$

For generic values of the mass m the conformal anomaly can be computed from expansions of $\Gamma_{\tau,\tau'}$ and $\Upsilon_{\tau,\tau'}$ in powers of D. These expansions both begin with terms of order D. Using these in (5.42) and (5.43) gives vanishing contributions to the anomaly. There are also contributions from $\mathrm{tr}\,hQ$ which cancel between (5.42) and (5.43). So for generic values of the mass the conformal anomaly is zero. But due to the divergent nature of $\Upsilon_{\tau,0}$ as $2|m|\to2\tilde{N}-1$ for any positive integer $\tilde{N}$, we have $\Upsilon_{\tau,\tau'}\to-4\Gamma^{-1}_{\tau',0}$, and this has a leading order term proportional to $1/D$ which combines with the D in h to give a finite contribution as $\xi\to0$. We conclude that the conformal anomaly is zero unless $2|m|$ is an odd integer $2\tilde{N}-1$. For a four-dimensional boundary these are precisely the values appearing in the mass spectrum of Supergravity compactified on $AdS_5\times S^5$ [19], and for a six-dimensional boundary they coincide with the mass spectrum of Supergravity compactified on $AdS_7 \times S^4$ [20]. In this case we have

$$\delta F = -\delta\rho \tilde{N} \frac{1}{(4\pi i)^N} \operatorname{tr} \int d^{2N}x \sqrt{g} Q_- a_N \,. \tag{5.49}$$

For $\mathbf{d} = 2$ and $r \to \infty$ the anomaly is proportional to $\tilde{a}_1 = -iR1/12$, so for a fermion with σ spinor components,

$$\delta F = \frac{\delta\rho}{96\pi} \int d^2x \sqrt{g} R \tilde{N} \sigma \,, \tag{5.50}$$

from which we identify the central charge as $\tilde{N}$.

For $\mathbf{d} = 4$ we have

$$\delta F = \frac{\delta\rho}{64\pi^2} \int d^4x \sqrt{g} \operatorname{tr} \tilde{a}_2(x) \tilde{N} \,, \tag{5.51}$$

where

$$\operatorname{tr} \tilde{a}_2 = \frac{\sigma}{720} \left(\frac{5}{2} R^2 + 6\Box R - \frac{7}{2} R_{ijkl} R^{ijkl} - 4 R_{ij} R^{ij} \right). \tag{5.52}$$

6. Gravitational sector

The tree-level conformal anomaly of pure gravity has been calculated for AdS_3 using the Chern-Simons formulation, [8, 9]. In higher dimensions it has been computed by solving the Einstein equations perturbatively in terms of boundary data, [10]. We will now show how our Schrödinger functional technique reproduces these results in a simple fashion. Our method employs a somewhat different regularization procedure, and so it is important to show that it is consistent with earlier calculations. The main difference is that authors of [10] effectively compute the classical free energy of the gravitational sector by finding $W_{\tau,\tau'}$ for $\tau = \infty$ and τ' a small regulator, whereas we take $\tau' = 0$ and treat τ as a large regulator. (Since we will work at tree-level it is not necessary to keep τ' nonzero as we did in the earlier computations of one-loop anomalies).

Formally we need to compute the euclidean functional integral that represents a state of pure gravity, with Einstein-Hilbert action and cosmological constant $\Lambda < 0$,

$$Z[g_{rs}] = \int \mathcal{D}G \exp\left(- \int d^{d+1}x \sqrt{G}(R + 2\Lambda) + \text{boundary terms} \right), \tag{6.1}$$

where we should integrate over all metrics $G_{\mu\nu}$ of a $d+1$ dimensional manifold which induce the metric g_{rs} on the boundary. This is ill-defined for a variety of reasons, such as non-renormalizability and unboundedness of the action, which we ignore in the hope that these pathologies are absent from the more fundamental theory of which this is only a part. The integral over all metrics includes reparametrizations which can be factored out using the Faddeev-Popov method. The standard ADM decomposition of the metric is

$$(G_{\mu\nu}) = \begin{pmatrix} N^2 + N_i N_j G^{ij} & N_j \\ N_i & G_{ij} \end{pmatrix} \tag{6.2}$$

with G^{ij} the inverse of the $d \times d$ matrix G_{ij}. We will fix the gauge by choosing

$$N^2 = \frac{L^2}{t^2}\,, \qquad N_i = 0\,, \qquad G_{ij} = \frac{g_{ij} + h_{ij}}{t^2} \tag{6.3}$$

with $t = x^0$. The dynamical variables are just the h_{ij}, and we take the boundary to be at $t = \tau = 0$, where $h_{ij} = 0$. The gauge conditions should be accompanied by the introduction of ghosts, but these will not contribute to the tree-level conformal anomaly. Expanding the action in powers of h_{ij}, and taking $L^2 = -d(d-1)/(2\Lambda)$ gives

$$\int d^{d+1}x\sqrt{G}(R + 2\Lambda) - \text{boundary terms} =$$
$$= \int d^{d+1}x \frac{\sqrt{g}L}{t^{d-1}} \left(R(g) + h_{ij}\tilde{G}^{ij}(g) + \frac{\dot{h}_{ij}\dot{h}^{ij} - (\dot{h}^i_i)^2}{4L^2} + h_{ij}\Box^{ijkl}h_{kl} + \cdots \right), \tag{6.4}$$

where the boundary terms are chosen to make the action quadratic in first derivatives of h_{ij}, and the dots denote terms of higher order in h_{ij}. Indices are raised and lowered with g_{ij}. $R(g)$ is the d-dimensional curvature calculated by taking g_{ij} as metric and $\sqrt{g}(R(g) + h_{ij}\tilde{G}ij + h_{ij}\Box^{ijkl}h_{kl})$ are the first three terms in the expansion of $\sqrt{det(g+h)}R(g+h)$, so that $\Box$ is a second order differential operator. The terms of higher order in h each contain one or two derivatives. As in section 1 the state $Z[g_{ij}]$ is the $\tau \to \infty$ limit of the Schrödinger functional. The the tree-level contribution to $\log Z[g_{rs}]$ is thus the $\tau \to \infty$ limit of minus the action evaluated on shell for a manifold with boundaries at $t = 0$, where the induced metric is g_{ij}, and $t = \tau$ where it is $g_{ij} + h_{ij}$. If we denote this as $W^{\text{tree}}_{\tau,0}[g_{ij} + h_{ij}, g_{ij}]$ then it satisfies the Hamilton-Jacobi equation, (which is the tree-level Schrödinger equation). This is simply the statement that $W^{\text{tree}}_{\tau+\delta\tau,0}$ can be obtained from $W^{\text{tree}}_{\tau,0}$ by allowing the fields to propagate according to the equations of motion from τ to $\tau + \delta\tau$, i.e.,

$$W^{\text{tree}}_{\tau+\delta\tau,0}\Big[g_{ij} + h_{ij} + \delta\tau\dot{h}_{ij}, g_{ij}\Big] = W^{\text{tree}}_{\tau,0}\Big[g_{ij} + h_{ij}, g_{ij}\Big] - L\delta\tau \tag{6.5}$$

where L is the lagrangian in (6.4). Using

$$-\frac{\delta W^{\text{tree}}_{\tau,0}}{\delta h_{ij}} = \frac{\sqrt{g}}{2L\tau^{d-1}}\left(\dot{h}^{ij} - g^{ij}\dot{h}^r_r\right) \equiv \pi^{ij} \tag{6.6}$$

this leads to

$$-\frac{\partial W^{\rm tree}_{\tau,0}}{\partial\tau} = L + \int d^dx \dot{h}_{ij}\frac{\delta W^{\rm tree}_{\tau,0}}{\delta h_{ij}} = \int d^dx\,\frac{\sqrt{g}L}{\tau^{d-1}}R(g) +$$
$$+\int d^dx\Big(-\frac{\tau^{d-1}L}{\sqrt{g}}\pi^{ij}G_{ijrs}\pi^{rs} + \frac{\sqrt{g}L}{\tau^{d-1}}\Big(h_{ij}\tilde{G}^{ij}(g) + h_{ij}\Box^{ijkl}h_{kl}\Big)\Big) + \cdots \tag{6.7}$$

with

$$G_{ijrs} = g_{ir}g_{js} - \frac{1}{d-1}g_{ij}g_{rs}\,. \tag{6.8}$$

and the initial condition is that $\exp W^{\rm tree}_{\tau,0}[\tilde{g}_{ij}, g_{rs}] \sim \delta[h_{ij}]$. When the curvature tensors constructed from g_{ij} are small, and h_{ij} is slowly varying we can expand $W^{\rm tree}$ in powers of h and its derivatives as

$$W^{\rm tree}_{\tau,0}\Big[g_{ij}+h_{ij}, g_{rs}\Big] =$$
$$= \int\frac{d^dx\sqrt{g}}{\tau^d}\Big(-\frac{d}{4L}h_{ij}G^{-1ijrs}h_{rs} + \Gamma_0 + +\Gamma_1^{ij}h_{ij} + h_{ij}\Gamma_2^{ijrs}h_{rs} + \cdots\Big). \tag{6.9}$$

Apart from a constant term in Γ_0, only the first term on the right has no derivatives of either h_{ij} or g_{ij}, and so provides the dominant behaviour as $\tau \to 0$, satisfying the initial condition (provided h^i_i is suitably treated, [11]). Substituting into (6.7) and equating powers of h_{ij} gives

$$-\frac{\partial}{\partial\tau}\left(\frac{\Gamma_0}{\tau^d}\right) = \frac{1}{\tau^{d-1}}LR(g) - \frac{L}{\tau^{d+1}}\Gamma_1^{ij}G_{ijrs}\Gamma_1^{rs}\,, \tag{6.10}$$

The free energy is the infinite τ limit of the h_{ij} independent part of (6.9), i.e., $F[\tau, g_{ij}] = \int d^dx\sqrt{g}\Gamma_0/\tau^d$, and, as before, a Weyl scaling of the metric can be compensated by a scaling of τ, so for $\delta g_{ij} = \delta\rho g_{ij}$ the change in F is given by (6.10)

$$\delta F = -\frac{\delta\rho}{2}\tau\frac{\partial F}{\partial\tau} = \frac{\delta\rho}{2}\int d^dx\sqrt{g}L\left(\frac{1}{\tau^{d-2}}R(g) - \frac{1}{\tau^d}\Gamma_1^{ij}G_{ijrs}\Gamma_1^{rs}\right). \tag{6.11}$$

Up to terms involving two derivatives, we have from (6.7)

$$-\frac{\partial\Gamma_1^{ij}}{\partial\tau} = L\tau\tilde{G}^{ij}(g)\,. \tag{6.12}$$

Solving this for Γ_1 and substituting into (6.11) gives the variation as an expansion in powers of τ times the curvatures constructed from g_{ij}. We want to work at finite τ, so our expansion will be valid when we take the curvatures to zero.

Now for $\mathbf{d} = 2$ we have identically $\tilde{G}^{ij} = 0$, so $\Gamma_1 = 0$ and to the order we need (6.11) reduces to

$$\delta F = \delta\rho\int d^2x\sqrt{g}L\frac{1}{2}R(g) \tag{6.13}$$

from which we can identify the central charge as $c = 24\pi L$, or, since we have chosen units such that the three-dimensional gravitational constant satisfies $16\pi G_{\mathrm{Newton}} = 1$, we have $c = 3L/(2G_{\mathrm{Newton}})$ as in [8].

For $\mathbf{d} = 4$ we obtain Γ_1 to the desired order from (6.12) as $\Gamma_1 = -L\tau^2\tilde{G}^{ij}/2$, so if we substitute into (6.10) we have

$$\delta F = \frac{\delta\rho}{2}\int d^4x\sqrt{g}L\left(\frac{R(g)}{\tau^2} - L^2\tilde{G}^{ij}G_{ijrs}\frac{\tilde{G}^{rs}}{4}\right). \tag{6.14}$$

The first term represents a divergence that should be cancelled by counter-terms, the last piece is the finite Weyl anomaly. If we reinstate the five-dimensional Newton constant this becomes

$$\delta\rho\frac{L^3}{128\pi G_{\mathrm{Newton}}}\left(R_{ij}R^{ij} - \frac{R^2}{3}\right) \tag{6.15}$$

which agrees with [10].

The results of this section can be generalized to higher-derivative gravity, as considered in [21].

7. Conclusions

We have interpreted the partition function for fields in euclideanized anti de Sitter space-time as a limit of the Schrödinger functional. This allows canonical methods to be used, in contrast to the usual approach based on Green's functions. The former have the advantage of separating out the time-dependence, and as a consequence we have been able to compute one-loop effects in the AdS theory by solving the functional Schrödinger equation, (as well as reproducing known tree-level results such as the scaling of two-point functions and the gravitational conformal anomaly). Although we have only studied the simplest one-loop quantity, namely the anomalous scaling of the free energy for scalar and fermionic theories, these canonical methods can be expected to apply to more complicated objects such as n-point functions, as they do in flat space [5, 6, 7].

Our computation of the one-loop conformal anomalies for scalar and fermionic theories shows that for generic values of the 'mass' the anomalies vanish, but for special values the anomalies have integer coefficients. For example, when the boundary of the AdS spacetime is two-dimensional the Virasoro central charges are positive integers, $\tilde{N}$, when the mass of the scalar field is $\sqrt{\tilde{N}^2 - 1}$ and when the mass of the fermion is $\tilde{N} - 1/2$. This implies that for generic values of the mass, the boundary CFTs for the scalar or fermi fields alone are nonunitary, since in a unitary theory a vanishing central charge implies an absence of quasi-primaries. When the boundary is four dimensional the conditions on the scalar and fermion masses for the conformal anomaly to be nonzero coincide with the mass spectra resulting from Kaluza-Klein compactification of Supergravity on $AdS_5 \times S^5$, and when it is six dimensional they coincide with the mass spectra resulting from Kaluza-Klein compactification of Supergravity on $AdS_7 \times S^4$. We expect that the same is true for the other fields in the Supergravity theory, and this would make the one-loop corrections to the tree-level calculation of [10] that checks the Maldacena conjecture an intriguing prospect.

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